Algorithmic Aspects of Congestion Games

Invited talk for

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Why Game Theory?

- Goal of TCS (1950-2000)?

  Develop a mathematical understanding of the capabilities and limitations of the **von Neumann computer** and its software —— the dominant and most novel computational artifacts of that time

- Today?

  Internet has surpassed the von Neumann computer as the most complex computational artifact of our time [Papadimitriou 2001]

- But:

  Internet is built, operated and used by a multitude of diverse (possibly conflicting) **economic interests** —— theoretical understanding urgently needed tools

- For example:

  Of which game is the TCP/IP protocol a stable state?

Game Theory vs TCS

- Game Theory

  A general theory studying the behavior of rational players.

- Implicit use of Game Theory in TCS

  ✓ Proving algorithmic bounds

  ✓ Online algorithms

  ✓ Learning

  ✓ Adversaries

  ✓ PSPACE

Non-cooperative Games

- **Strategic Game:** $(N, (\Pi_i)_{i \in N}, (U_i)_{i \in N})$ where $\forall i \in N$, $U_i : \times_{i \in N} \Pi_i \mapsto \mathbb{R}$ is user i's utility function.

- **Pure Strategies:** Each user i chooses an action from its action set action set Π_i with certainty.

- **Mixed Strategies:** Each user i chooses a probability distribution over its action set Π_i.

- N, Π_i are considered to be finite here.

What is a Rational Behavior in a Game?

- Nash Equilibrium: A combination of strategies for the users so that no user has the incentive to change *unilaterally* its own strategy [Nash, 1951] $\Rightarrow$ ALWAYS EXISTS!

- Problems:
  - A Pure NE may not exist $\Rightarrow$ DECIDABILITY
  - A mixed NE always exists $\Rightarrow$ COMPUTABILITY ($\in P$?)
  - Many NE may exist $\Rightarrow$ WHICH IS BEST?

Price of Anarchy

- **Approximation ratio:** Price for not having exponential resources.
- **Competitive ratio:** Price for not knowing the future.
- **Coordination ratio:** Price for not having coordination (due to selfish players) – also called **price of anarchy**.

Price of Anarchy (contd.)

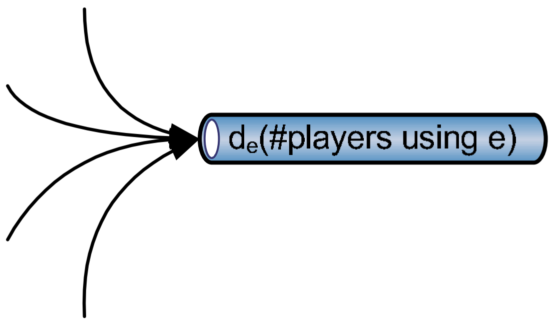
[Koutsoupias, Papadimitriou, 1999]

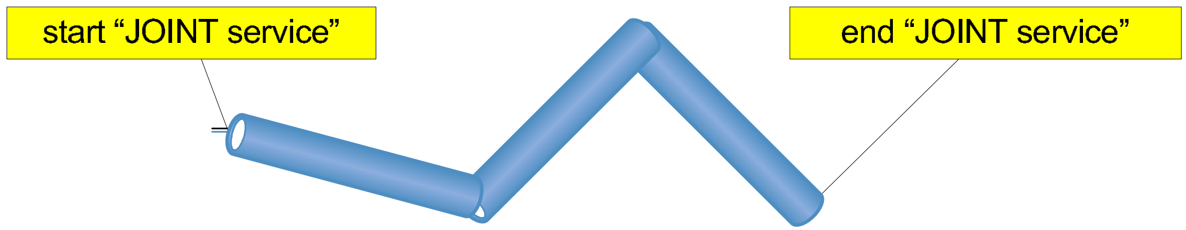
$$\mathcal{R} = \frac{\text{social cost of Worst NE}}{\text{optimum cost}}$$

- **Social Cost** = a global (system) measure of performance (eg, max delay in traffic).
- $\mathcal{R} \geq 1$

Congestion Games

[Rosenthal 1973, Monderer & Shapley 1996]





- A set E of shared resources.
- A set N of non-cooperative players with *identical* demands ($\forall i \in N,\ w_i = 1$).
- $\forall i \in N,\ \Pi_i \subseteq 2^E \setminus \emptyset$ is the set of allowable actions for player i (action = a non-empty collection of resources).
- Each resource $e \in E$ has a non-decreasing delay function $d_e : \mathbb{R}_+ \mapsto \mathbb{R}_+$, depending only on the *cumulative congestion* (ie, #players using the same resource).

Congestion Games (contd.)

- Wrt to a given *pure strategies profile* $\varpi \in \times_{i \in N} \Pi_i$, the selfish cost of player i taking action $\varpi_i \in \Pi_i$ is:

$$\boxed{\lambda^i(\varpi) = \lambda_{\varpi_i}(\varpi) = \sum_{e \in \varpi_i} d_e(\theta_e(\varpi))}$$

where,

$$\boxed{\Lambda_e(\varpi) \equiv \{i \in N : e \in \varpi_i\}}$$

is the set of players using resource e according to ϖ, and

$$\boxed{\theta_e(\varpi) \equiv \sum_{i \in \Lambda_e(\varpi)} w_i}$$

is the total load on resource e wrt ϖ.

Congestion Games (contd.)

- ϖ^{-i}: a configuration of all players except i.
- $\mathbf{p}^{-i}$: the mixed strategies profile of all players except i.
- $\varpi^{-i} \oplus \varpi_i$: the new configuration with player i choosing the pure strategy ϖ_i.
- $\mathbb{P}[A]$: the probability of event A occurring.
- $P(\mathbf{p}, \varpi) = \prod_{i \in N} p_i(\varpi_i)$: the probability of configuration ϖ occurring, when the players adopt the mixed profile $\mathbf{p}$.

Congestion Games (contd.)

- Wrt to a given *mixed strategies profile* $\mathbf{p}$, the selfish cost of player i taking action $\varpi_i \in \Pi_i$ is the expectation of the respective random variable [von Neumann & Morgenstern 1944]:

$$\lambda^i_{\varpi_i}(\mathbf{p}) = \sum_{\varpi^{-i} \in \Pi^{-i}} P(\mathbf{p}^{-i}, \varpi^{-i}) \cdot \sum_{e \in \varpi_i} d_e \left(\theta_e(\varpi^{-i} \oplus \varpi_i)\right)$$

- **Social Cost** of a mixed strategies profile $\mathbf{p}$:

$$SC(\mathbf{p}) = \sum_{\varpi \in \Pi} P(\mathbf{p}, \varpi) \cdot \max_{i \in N} \{\lambda_{\varpi_i}(\varpi)\}$$

- **Social Optimum:** $OPT = \min_{\varpi \in \Pi} \{\max_{i \in N} [\lambda_{\varpi_i}(\varpi)]\}$

- NOTE: $\max_{i \in N}$ may be replaced by some other computable function of N.

- **Price of Anarchy:** $\mathcal{R} = \max_{\mathbf{p} \text{ is a NE}} \left\{ \frac{SC(\mathbf{p})}{OPT} \right\}$

Categories of Congestion Games

A congestion game is...

- symmetric, if all players are *indistinguishable* (ie, have the same action set and the same utility function).

- a (multi-commodity) network congestion game, if for each user i, its allowable actions are (s_i, t_i)-paths in the graph of the resources.

- a single-commodity network congestion game if all allowable actions of the players are (s, t)-paths in the graph of resources.

Potential Games

[Monderer & Shapley 1996]

- $\Gamma = (\Pi_i,\ U_i : \Pi \mapsto \mathbb{R})_{i \in N}$: A strategic game, where $\Pi \equiv \times_{i=1}\Pi_i$ is the set of possible pure strategies profiles.

- Neighboring Pure Profiles: $\forall \varpi \in \Pi,\ \forall i \in N,\ \forall z_i \in \Pi \setminus \{\varpi_i\}$,

$$\boxed{\varpi^{-i} \oplus z_i \equiv (\varpi_1, \varpi_2, \ldots, \varpi_{i-1}, z_i, \varpi_{i+1}, \ldots, \varpi_n)}$$

  ϖ and $\varpi^{-i} \oplus z_i$ are neighboring pure profiles.

Potential Games (contd.)

RA CTI

- For the given game Γ, a function $\Phi : \Pi \mapsto \mathbb{R}$ is
  - an ordinal potential iff $\forall i \in N, \forall \varpi \in \Pi, \forall z_i \in \Pi_i,$

    $$U_i(\varpi) - U_i(\varpi^{-i} \oplus z_i) > 0 \Leftrightarrow \Phi(\varpi) - \Phi(\varpi^{-i} \oplus z_i) > 0$$

  - a **b**-potential, iff $\forall i \in N, \forall \varpi \in \Pi, \forall z_i \in \Pi_i,$

    $$U_i(\varpi) - U_i(\varpi^{-i} \oplus z_i) = b_i \cdot \left(\Phi(\varpi) - \Phi(\varpi^{-i} \oplus z_i)\right)$$

  - an exact potential, iff it is a **1**-potential.

Properties of Potential Games

[Monderer & Shapley 1996]:

- A path in Π is a sequence of configurations $\gamma = \langle \varpi(0), \varpi(1), \ldots \rangle$ such that $\forall k \geq 1$ there exists a unique player i_k such that $\varpi(k) = \varpi(k-1)^{-i} \oplus \pi_i$ for some action $\pi_i \in \Pi_i \setminus \{\varpi(k-1)_i\}$.

- γ is an improvement path if $\forall k \geq 1,\ U_i(\varpi(k)) > U_i(\varpi(k-1))$ where i_k is the unique deviator at step k.

Definition 1 *A game has the Finite Improvement Property (FIP) if every improvement path has finite length.*

Properties of Potential Games (contd.)

Theorem 2 [Monderer & Shapley 1996]
Every finite ordinal potential game has the Finite Improvement Property.

Corollary 3 *Every finite ordinal potential game has at least one Pure Nash Equilibrium.*

Properties of Potential Games (contd.)

- Fictitious Play: When the best response dynamics converges, starting from arbitrary mixed strategies profile.

Theorem 4 *Every finite* $\mathbf{b}$*-potential game has the Fictitious Play property.*

The Potential/Congestion Theorem

Main Theorem 5 [Monderer & Shapley 1996]

(a) *Every congestion game is an exact potential game.*

(b) *Every (finite) potential game is isomorphic to a congestion game.*

Change of Utilities of Deviators in Closed Paths

- For a finite path $\gamma = \langle \varpi(0), \ldots, \varpi(k) \rangle$ and a collection $U = \{U_i\}_{i \in N}$ of utility functions, define

$$I(\gamma, U) = \sum_{r=1}^{k} [U_{i_r}(\varpi(r)) - U_{i_r}(\varpi(r-1))]$$

where $\forall r \in [k],\ i_k$ is the **unique deviator** at step r.

Theorem 6 *Let Γ be a game in strategic form. The following are equivalent:*

(1) Γ is an exact potential game.

(2) $I(\gamma, U) = 0$ for every finite simple closed path γ of length 4.

From this point on, the results are of

[Fotakis, Kontogiannis, Spirakis 2004]

in ICALP 2004 and DELIS SP4/FET/EU

Weighted Congestion Games

[Fotakis, Kontogiannis, Spirakis 2004]

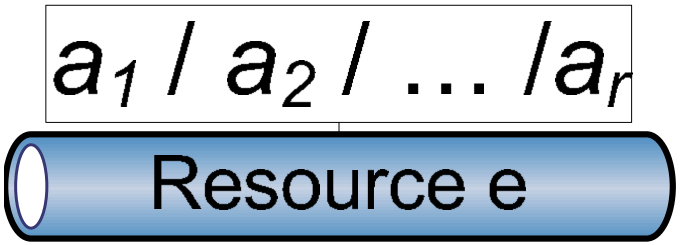
- Each player i has a non-negative weight w_i (traffic demand). The weights are non-identical:

$$\boxed{\mathbf{w} = (w_i)_{i \in N} \in \mathbb{R}_{\geq 0}^{|N|}}$$

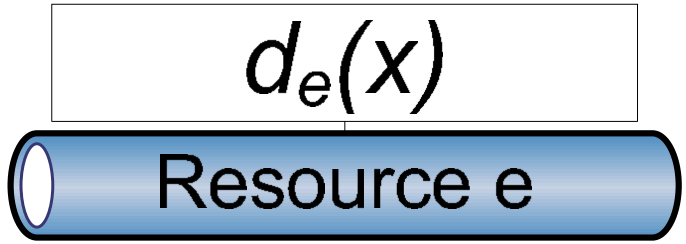
- Total load on resource e:

$$\boxed{\theta_e(\varpi) = \sum_{i \in \Lambda_e(\varpi)} w_i}$$

Some Notation



(A)



(B)

Meaning:

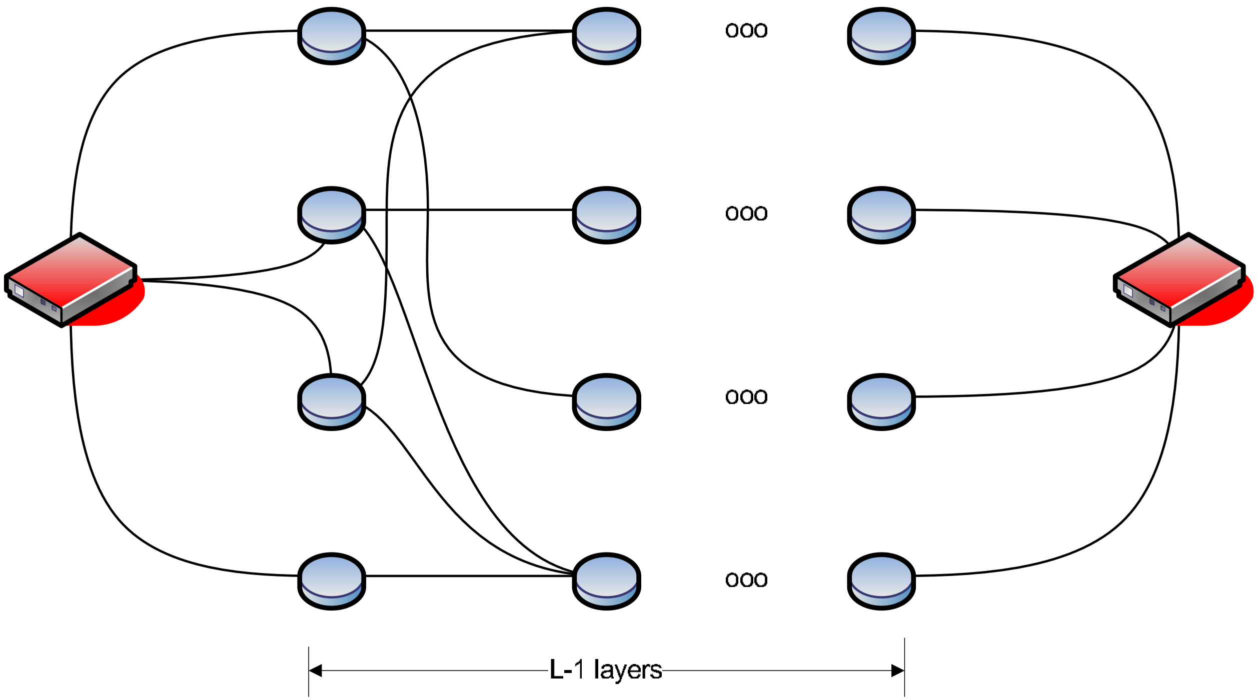
(A) r possible values of total load may appear in resource $e \in E$. For the k^{th} smallest load, the delay of e is a_k.

(B) A continuous function $d_e(x)$ determines the delay of resource e as a function of its load.

Resource Delay Functions:

- In general, non-decreasing functions of loads.
- Special cases: Linear delays and two-wise linear delays (ie, maximum of two linear functions).

Layered Networks



- All players want to route traffic from a unique source s to a unique destination t (single-commodity network).
- All the nodes of the network lie on an (s,t)-path.
- Edges (representing shared resources) can only exist between nodes of consecutive layers.
- Each (s,t)-path in the network has length exactly L.

$\exists$ PNEs in Weighted Congestion Games?

What we know:

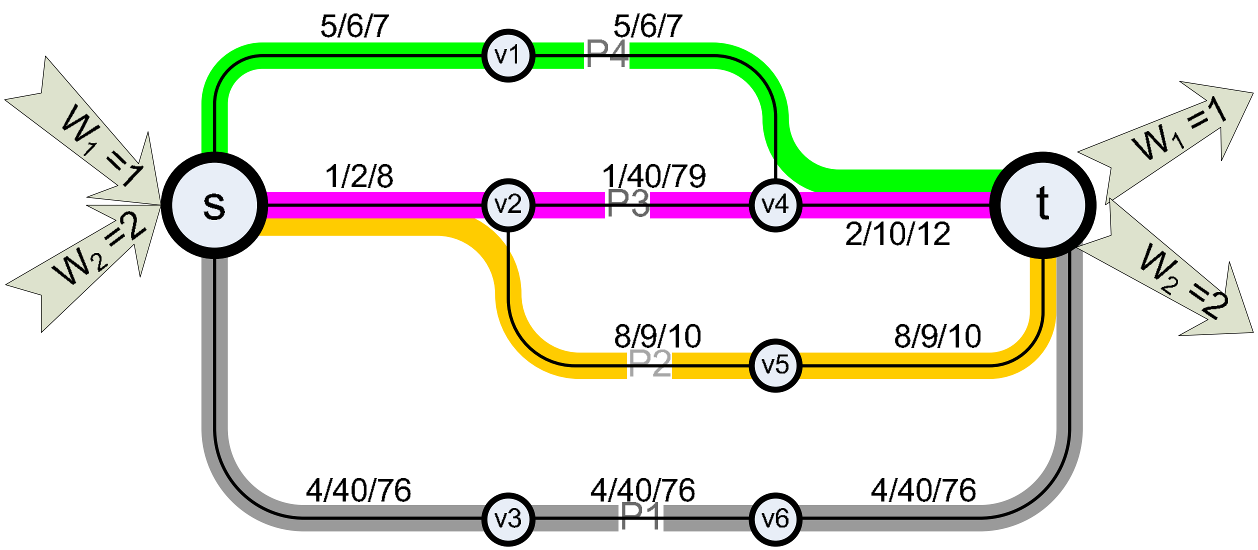
Theorem 7 [Rosenthal 1973]
Any (unweighted) congestion game has at least one Pure Nash Equilibrium.

Our Result

Theorem 8 [Fotakis, Kontogiannis, Spirakis 2004]
*Even a 3-layered network weighted congestion game with **2-wise linear resource delays** may have no PNE.*

Proof:

- $\exists$ 4-cycle $\langle (P3,P2),(P3,P4),(P1,P4),(P1,P2),(P3,P2)\rangle$ in the *Best Response Dynamics graph* of the game.
- Any pure strategies profile out of this cycle is either one or two best-response moves away from some of its configurations.



Exact Potentials for Weighted Congestion Games?

What we know:

Theorem 9 [Rosenthal 1973]
Every (unweighted) congestion game admits an exact potential.

Our Result

Theorem 10 [Fotakis, Kontogiannis, Spirakis 2004]
*Even a single commodity network congestion game with **resource delays equal to the congestions** may not have an exact potential.*

Proof:

- Cycle = a sequence of pure strategies profiles $\gamma = \langle \varpi(0), \ldots, \varpi(r) = \varpi(0) \rangle$, where $\forall k,\ i_k$ is the unique player in which $\varpi(k)$ and $\varpi(k-1)$ differ.

- [Monderer & Shapley 1996]: Let $I(\gamma) = \sum_{k=1}^{r} \left[\lambda^{i_k}(\varpi(k)) - \lambda^{i_k}(\varpi(k-1)) \right]$. A game admits an *exact potential* iff any 4-cycle γ has $I(\gamma) = 0$.

- The 4-cycle $\gamma = (\varpi,\ \varpi^{-1} \oplus \pi_1,\ \varpi^{-1,2} \oplus \{\pi_1, \pi_2\},\ \varpi^{-2} \oplus \pi_2,\ \varpi)$ has $I = (w_1 - w_2) \cdot$ NETWORK CONSTANT and is typically non-zero.

The Dynamics Graph

Definition 11 *The Dynamics Graph of a game Γ is a directed graph whose*

- *vertices are configurations of the players, and*
- $\forall \varpi \in \Pi$, $\forall i \in N$, $\forall \pi_i \in \Pi_i \setminus \{\varpi_i\}$, *there is an arc from* ϖ *to* $\varpi^{-i} \oplus \pi_i$ *iff*

$$\boxed{\lambda^i(\varpi) > \lambda^i(\varpi^{-i} \oplus \pi_i)}$$

Construction of a PNE in Congestion Games?

What we know:

Theorem 12 [Fabrikant, Papadimitriou, Talwar 2004]

- *There is a polynomial time algorithm for finding a Pure Nash Equilibrium in **symmetric network congestion games**.*
- *It is PLS-complete to find a Pure Nash Equilibrium even for **asymmetric network congestion games**.*

Our Result

Theorem 13 [Fotakis, Kontogiannis, Spirakis 2004]
For any weighted L-layered network congestion game with resource delays equal to their congestions, at least one PNE exists and can be constructed in time $\frac{1}{2}|E|W_{tot}^2$.

Proof:

- $\Phi(\varpi) = \sum_{e \in E} [\theta_e(\varpi)]^2$ is a $\left(\frac{1}{2w_i}\right)_{i \in N}$-potential for the game.
- Wlog assume that players have *integer* weights.
- Each arc in the Dynamics Graph decreases the potential by at least at least $2w_{\min} \geq 2$.

$\Rightarrow$ Any improvement path has length at most $\frac{1}{2}|E|W_{\text{tot}}^2$.

An Improvement

For any weighted congestion on an arbitrary *multi-commodity network* whose resource delays are *linear* functions of their loads (ie, $\forall e \in E, \ d_e(x) = a_e \cdot x + b_e$), we can construct a Pure Nash Equilibrium in pseudo-polynomial time.

[Fotakis, Kontogiannis, Spirakis 2004]

What about the Price of Anarchy?

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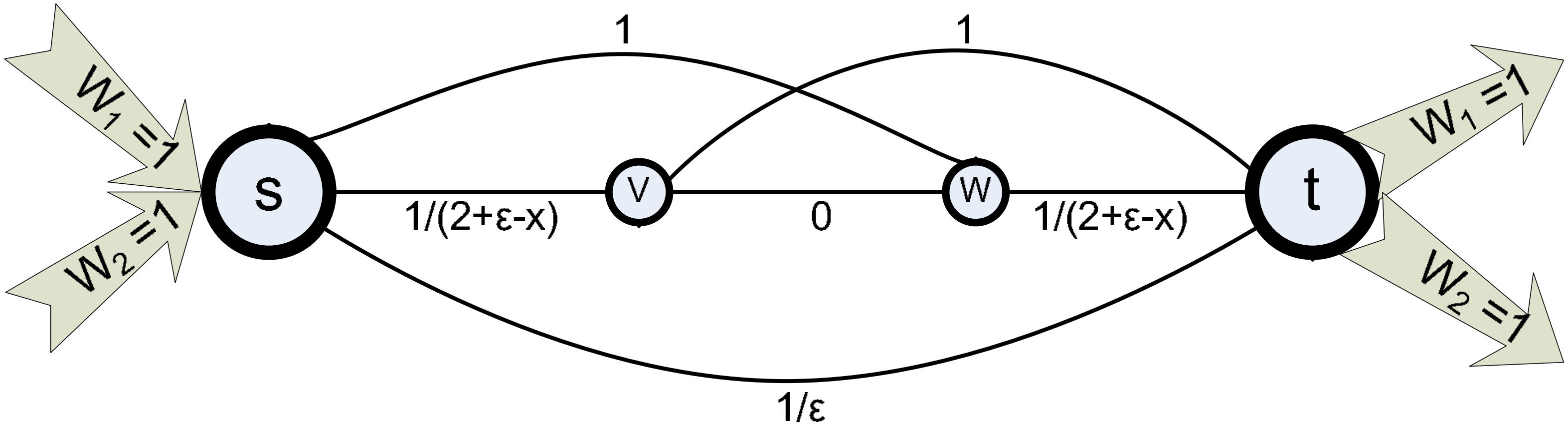
The price of anarchy can be unbounded, even in *unweighted* layered network congestion games with *linear* resource delays.

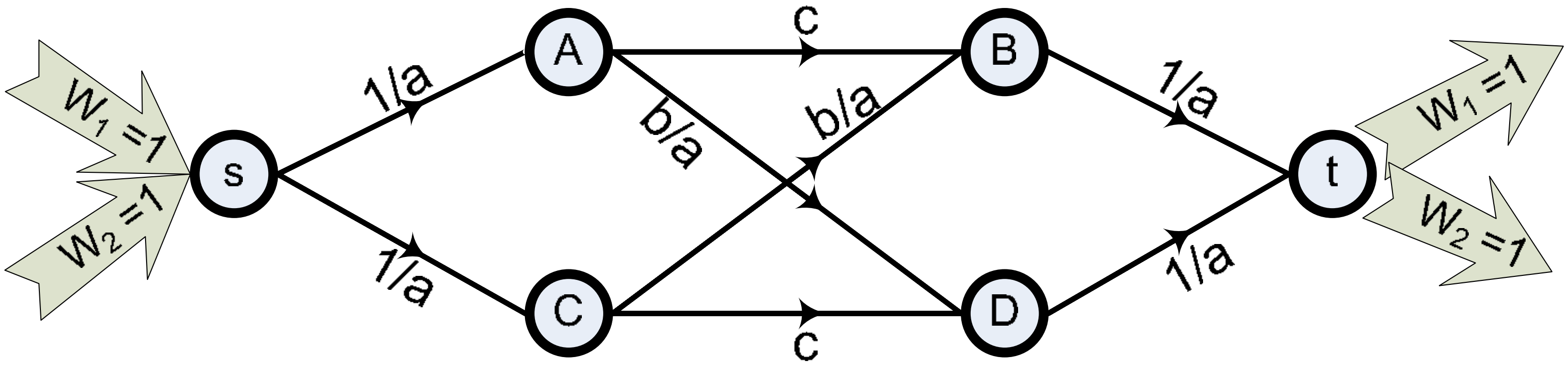
Anarchy of Network Congestion Games

Example of [Roughgarden, Tardos 2000] for atomic flows (can easily be transformed into a 3-layered network congestion game):



- Identical users.
- Constant and M/M/1-like resource delays.
- **OPT**:(svt, swt)
- **NASH**: (st, svwt)
- $\mathcal{R} = \frac{1+\varepsilon}{(2+\varepsilon)\cdot\varepsilon}$

Anarchy of Layered Networks with Linear Delays



- Identical players.
- *Linear* resource delays.
- $a \gg b \gg 1 \geq c \geq 0$.
- **OPT**:(sABt,sCDt)
- **NASH**: (sADt, sCBt)
- $\mathcal{R} = \frac{2+b}{2+c}$

What Remains?

We seek for

(I) the price of anarchy of *layered-network weighted congestion games* with resource delays *proportional* to their loads, or

(II) the price of anarchy of *unweighted* congestion games on *general single-commodity networks*.

(I) Unit-Weight Players

- Delays proportional to resource loads:

$$\forall e \in E,\ d_e(x) = a_e \cdot x :\ a_e \geq 0$$

- General (single-commodity) network.

- m = #edges.

The Network $G = (V, E)$

- P: the set of all (simple) paths from the unique source $s \in V$ to the unique destination $t \in V$.
- $\forall e \in E,\ d_e(x) = a_e \cdot x :\ a_e \geq 0$
- $m = |E|$
- $\forall \pi \in P,\ a_\pi \equiv \sum_{e \in \pi} a_e$
- Feasible Flow: A function $\rho : P \mapsto \mathbb{R}_{\geq 0}$ s.t. $\sum_{\pi \in P} \rho(\pi) = n$ (all player demands are met)
- NOTE: n is both the number of players and the total demand that has to be routed from s to t.

Flows on G

- Unsplittable Flow: Each player's demand is routed via a **unique** s-t path.

- Splittable Flow: The demand of each player can be split over several s-t paths.

Mapping Mixed Profiles to Flows

- $\mathbf{p} = (p_1, \ldots, p_n)$ is an arbitrary mixed strategies profile.

- We map $\mathbf{p}$ to the (splittable) flow $\rho_{\mathbf{p}}$ which is defined as follows:

$$\forall \pi \in P, \quad \rho_{\mathbf{p}}(\pi) = \sum_{i \in [n]} p_i(\pi)$$

(ie, the **expected load** travelling along π is viewed as a splittable flow on this path).

- NOTE: If $\mathbf{p}$ is a pure strategies profile then the corresponding flow is unsplittable.

Flow Latencies vs Expected Delays

- The expected delay of resource $e \in E$ wrt the profile $\mathbf{p}$ is

$$\boxed{\theta_e(\mathbf{p}) \equiv a_e \cdot \sum_{i\in[n]} \sum_{\pi \ni e} p_i(\pi) = a_e \cdot \rho_{\mathbf{p}}(e) \equiv \theta_e(\rho_{\mathbf{p}})}$$

(ie, the expected delay of a resource wrt to a mixed profile is equal to the latency caused by the corresponding flow).

$\Rightarrow$ $\theta_\pi(\mathbf{p}) = \sum_{e\in\pi} \theta_e(\mathbf{p}) = \theta_\pi(\rho_{\mathbf{p}})$

(ie, the expected delay along a path wrt to a mixed profile is equal to the total latency on this path caused by the corresponding flow).

- Maximum Latency of a flow $\rho = \rho_{\mathbf{p}}$:

$$\boxed{L(\rho) \equiv \max_{\pi:\rho(\pi)>0} \{\theta_\pi(\rho)\} = \max_{\pi:\exists i\in[n], p_i(\pi)>0} \{\theta_\pi(\mathbf{p})\} \equiv L(\mathbf{p})}$$

(ie, the maximum latency caused by the flow $\rho = \rho_{\mathbf{p}}$ is equal to the maximum expected delay paid by the users wrt to the mixed strategies profile $\mathbf{p}$).

Alternative Measures of Flows

RA CTI

- Total Latency:

$$C(\rho) \equiv \sum_{\pi \in P} \rho(\pi)\theta_\pi(\rho) = \sum_{e \in E} a_e \rho^2(e) \equiv C(\mathbf{p})$$

- Total Load:

$$W(\rho) \equiv \sum_{e \in E} a_e \rho^2(e) = \sum_{\pi \in P} a_\pi \rho(\pi) \equiv W(\mathbf{p})$$

- For any feasible flow ρ let
  - $a(\rho) \equiv \max_{\pi:\rho(\pi)>0}\{a_\pi\}$ and
  - $d^{min}(\rho) \equiv \min_{\pi \in P}\{\theta_\pi(\rho) + a_\pi\}$

Flows at Nash Equilibrium

- $\mathbf{p}$: an arbitrary mixed strategies profile and $\rho_{\mathbf{p}}$ is the corresponding flow.
- The (exprected) cost of player $i \in [n]$ for using path $\pi \in P$ is

$$\boxed{\lambda_\pi^i(\mathbf{p}) = \theta_\pi^{-i}(\mathbf{p}) + a_\pi}$$

where $\theta_\pi^{-i}(\mathbf{p}) = \theta_\pi(\mathbf{p}) - \sum_{\pi' \in P} Q[\pi, \pi'] p_i(\pi')$ is the expected delay along path π caused by all players except for player i.

Definition 14 *The flow $\rho = \rho_{\mathbf{p}}$ is at Nash equilibrium iff the corresponding mixed profile $\mathbf{p}$ is at Nash equilibrium.*

Proposition 15 *If ρ is a Nash flow then $\forall \pi \in P: \ \rho(\pi) > 0,$*

$$\boxed{\max\{\theta_\pi(\rho), a_\pi\} \leq d^{min}(\rho) \equiv \min_{\pi' \in P}\{\theta_{\pi'}(\rho) + a_{\pi'}\}}$$

Notation

- $\forall \pi, \pi' \in P,\ Q[\pi, \pi'] \equiv \sum_{e \in \pi \bigcap \pi'} a_e$ (a $|P| \times |P|$ *symmetric* matrix).

- $C(\rho) = \rho^T Q \rho = \sum_{e \in E} a_e \rho^2(e) \geq 0,\ \forall \rho \in \mathbb{R}^{|P|}$

$\Rightarrow$ Q is a *positive semidefinite* matrix.

- $\forall \pi \in P,\ A[\pi] \equiv a_\pi = \sum_{e \in \pi} a_e$

- $W(\rho) = A^T \rho,\ \forall \rho.$

A Useful Quadratic Program

Proposition 16 *Let* ρ *be a Nash flow. Then,* $\forall a \in [0,1]$*,*

$$aC(\rho) + (1-a)W(\rho) \leq nd^{min}(\rho).$$

Definition 17 *Let* $\hat{\rho}$ *be the optimal (splittable) flow of the following quadratic program:*

$$\text{(QP1)} \quad \boxed{\min\left\{\tfrac{n-1}{2n}\rho^T Q\rho + A^T\rho : \mathbf{1}^T\rho \geq n; \quad \rho \geq \mathbf{0}\right\}}$$

Remark 18 *Here* n *denotes both the #players and the total traffic demand.*

A Mixed Nash Equilibrium

Proposition 19 *Let* $\mathbf{p}$ *be a mixed strategies profile where every player* $i \in [n]$ *routes its traffic on each path* $\pi \in P$ *with probability* $p_i = \frac{\hat{\rho}(\pi)}{n}$. *Then* $\mathbf{p}$ *is a Nash Equilibrium.*

Yet Another Quadratic Program

Definition 20 *Let $\bar{\rho}$ be the optimal (splittable) flow of the following quadratic program:*

$$\text{(QP2)} \quad \boxed{\min\left\{\rho^T\left(\tfrac{1}{2}Q\right)\rho + A^T\rho : \mathbf{1}^T\rho \geq n; \quad \rho \geq \mathbf{0}\right\}}$$

Remark 21 *Due to optimality of $\bar{\rho}$, it holds that $\sum_{\pi \in P} \bar{\rho}(\pi) = n$.*

Lemma 22 *For any feasible flow ρ corresponding to a mixed strategies profile at Nash equilibrium,*

$$C(\rho) + W(\rho) \leq 4\left[\tfrac{1}{2}C(\bar{\rho}) + W(\bar{\rho})\right].$$

Proof: Using Dorn's Theorem on strong duality of Quadratic programs.

Nash Flows vs Optimal Unsplittable Flow

Lemma 23 *Let ρ^* be the optimal unsplittable flow wrt the maximum latency objective. Then,*

$$\frac{1}{2}C(\overline{\rho}) + W(\overline{\rho}) \leq \frac{3}{2}C(\rho^*)$$

Main Lemma 24 *For any feasible flow ρ corresponding to a mixed strategies Nash equilibrium,*

$$\max\{L(\rho), a(\rho)\} \leq 6L(\rho^*)$$

Proof: By contradiction.

Price of Anarchy: $\mathcal{R} = \max_{\mathbf{p} \text{ is a NE}} \frac{SC(\mathbf{p})}{L(\rho^*)}$

Statistical Conflict

Lemma 25 *Let*

- ρ^*: *the optimal unsplittable flow wrt the max-latency objective.*
- $\rho = \rho_{\mathbf{p}}$: *the feasible flow corresponding to a mixed strategies profile* $\mathbf{p}$.

Assume that there is some constant $\beta \geq 1$ *s.t.* $\max\{L(\rho), a(\rho)\} \leq \beta \cdot L(\rho^*)$. *Then,*

$$\boxed{SC(\mathbf{p}) \leq 2\beta \cdot O\left(\frac{\ln n}{\ln \ln m}\right) \cdot L(\rho^*)}$$

Proof Sketch

- X_e: the r.v. for the actual delay on edge $e \in E$.
- Hoeffding bound:

$$\mathbb{P}\left[X_e \geq ek \max\{\theta_e(\rho), a_e\}\right] \leq k^{-ek}$$

- $X_\pi \equiv \sum_{e \in \pi} X_e$: the r.v. for the actual delay on path $\pi \in P$.
- Use the following facts:
  - $SC(\mathbf{p}) \leq \mathbb{E}\left[\max_{\pi:\rho(\pi)>0}\{X_\pi\}\right]$
  - $\mathbb{P}\left[\max_{\pi \in P}\{X_\pi\} \geq 2e\beta kL(\rho^*)\right] \leq mk^{-ek}$

The Bound on the Price of Anarchy

By lemmas 24 and 25 we conclude that

$$\boxed{\mathcal{R} \leq 24e\left(\frac{\ln m}{\ln \ln m} + 1\right)}$$

(II) Different Demands on Layered Networks

- Players have distinct weights.
- $\rho : P \mapsto \mathbb{R}_{\geq 0}$ is a feasible flow if $\sum_{\pi \in P} \rho(\pi) = W_{total} = \sum_{i \in [n]} w_i$.
- Mapping of (feasible) flows to mixed strategies profiles $\mathbf{p}$:

$$\forall \pi \in P,\ \rho_{\mathbf{p}}(\pi) = \sum_{i \in [n]} w_i \cdot p_i(\pi).$$

- For a Nash flow ρ, and the optimum unsplittable flow ρ^* wrt the max-latency objective,

$$L(\rho) \leq 3 \cdot L(\rho^*)$$

(again use Dorn's theorem).

Anarchy of Weighted Layered Network Games

For weighted congestion games on L-layered networks, the price of anarchy is

$$\boxed{\mathcal{R} \leq 8e\left(\frac{\ln m}{\ln\ln m} + 1\right)}$$

A Simplified Quadratic Program

The quadratic program we used in this case is

(QP3) $$\min \left\{ \rho^T Q \rho : \mathbf{1}^T \rho \geq W_{total}; \quad \rho \geq \mathbf{0} \right\}$$

and its dual is

(DP3) $$\max \left\{ z \cdot W_{total} - \rho^T Q \rho : 2Q\rho \geq \mathbf{1}z; z \geq 0 \right\}$$

from which (using Dorn's strong duality theorem) we get that

$$L(\rho) \leq 3 \cdot L(\rho^*)$$

Open Problems

RA CTI

- Weighted users **and** general network?
- Multicommodity congestion games?
- Polynomial time algorithm for construction of Pure Nash Equilibria (or PLS-Complete)?

Thank you!